

A 2-CALABI-YAU REALIZATION OF FINITE-TYPE CLUSTER ALGEBRAS WITH UNIVERSAL COEFFICIENTS

ALFREDO NÁJERA CHÁVEZ

ABSTRACT. We construct a 2-Calbi-Yau realization of various finite-type cluster algebras with coefficients using completed orbit categories associated to Frobenius categories. Namely, the Frobenius categories we consider are the categories of representable functors over (a quotient of) the Nakajima category associated to a Dynkin diagram. In particular, we are able to categorify all finite-type skew-symmetric cluster algebras with universal coefficients. Along the way, we classify the standard Frobenius models of certain family of triangulated orbit categories which include all finite-type n -cluster categories, for all integers $n \geq 1$.

1. INTRODUCTION

In [4], Buan, Marsh, Reineke, Reiten and Todorov introduced cluster categories. These are certain kind of triangulated categories which provide a categorification of cluster algebras associated to acyclic quivers. Let us recall their construction. Let k be an algebraically closed field and Q a finite quiver with no oriented cycles. The cluster category \mathcal{C}_Q is defined as the orbit category

$$\mathcal{D}^b(\text{mod}(kQ))/\Sigma \circ \tau^{-1},$$

where $\mathcal{D}^b(\text{mod}(kQ))$ is the bounded derived category of finite-dimensional right modules over the path algebra kQ and $\Sigma \circ \tau^{-1}$ is the composition of the suspension functor Σ and the inverse of the Auslander-Reiten translation τ^{-1} . Some of the fundamental properties of \mathcal{C}_Q are that it is a 2-Calabi-Yau Krull-Schmidt triangulated category with a cluster tilting subcategory.

Following [5], we can seek for stably 2-Calabi-Yau Krull-Schmidt Frobenius categories with a cluster tilting subcategory in order to categorify cluster algebras associated to ice quivers, also called cluster algebras of geometric type with coefficients or simply geometric cluster algebras. This approach has been used by many authors. We can mention for example Geiss-Leclerc-Schröer's categorification of the multi-homogenous coordinate ring of a partial flag variety [15], the categorification of Grassmannian cluster algebras by Jensen, King and Su [17], the recent generalization of [15] and [17] by Demonet and Iyama [7], the categorification of cluster algebras associated to ice quivers with potential arising from triangulated surfaces carried out by Demonet and Luo in [8], among many other works.

In this paper we combine the approaches of [4] and [5]. Let \mathcal{E} be a Frobenius model of $\mathcal{D}^b(\text{mod}(kQ))$ in the sense of [20], *i.e.* \mathcal{E} is a k -linear, **Hom**-finite Krull-Schmidt category endowed with the structure of a Frobenius category whose stable category is triangle equivalent to $\mathcal{D}^b(\text{mod}(kQ))$. Suppose that the autoequivalence $\Sigma \circ \tau^{-1}$ can be lifted to an exact autoequivalence $E : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$. By the work of [21], the *completed orbit category* \mathcal{E}/E becomes a Frobenius category whose stable category is triangle equivalent to the cluster category \mathcal{C}_Q . It is reasonable to expect that the Frobenius category \mathcal{E}/E leads to a categorification of a geometric cluster algebra of type Q . In this article we prove that this is indeed the case if Q is an orientation of a ADE Dynkin diagram and \mathcal{E} satisfies the following conditions:

- (i) For each indecomposable projective object P of \mathcal{E} , the \mathcal{E} -module $\text{rad}_{\mathcal{E}}(?, P)$ and the \mathcal{E}^{op} -module $\text{rad}_{\mathcal{E}}(P, ?)$ are finitely generated with simple tops.
- (ii) \mathcal{E} is standard in the sense of Ringel [30], *i.e.* its category of indecomposable objects is equivalent to the mesh category of its Auslander–Reiten quiver.

To make a rigorous statement let us first recall an important construction due to Keller and Scherotzke where conditions (i) and (ii) appear naturally. Suppose that the underlying graph of Q is a Dynkin diagram Δ . By the work of Happel [16], can identify the vertices of the repetition quiver $\mathbb{Z}Q$ with the set of indecomposable objects of $\mathcal{D}^b(\text{mod } kQ)$. It was proved in [20] that the Frobenius models of $\mathcal{D}^b(\text{mod}(kQ))$ satisfying conditions (i) and (ii) are in bijection with the set of *admissible configurations* of vertices of $\mathbb{Z}Q$ (see Definition 7). This bijection can be described as follows. Let \mathcal{R} be the Nakajima category (in the sense of [20]) associated to Q and $C \subset \mathbb{Z}Q$ an admissible configuration. Let \mathcal{R}_C be the quotient category of \mathcal{R} associated to C constructed in [20] (*cf.* Definition 7). Then the category of projective \mathcal{R}_C -modules $\text{proj}(\mathcal{R}_C)$ is a Frobenius model of $\mathcal{D}^b(\text{mod}(kQ))$ satisfying the conditions (i) and (ii). Moreover, every Frobenius model of $\mathcal{D}^b(\text{mod } kQ)$ satisfying these conditions is equivalent as an exact category to $\text{proj}(\mathcal{R}_C)$ for some admissible configuration. The main result of this paper is the following.

Theorem. Let C be an admissible configuration of vertices of $\mathbb{Z}Q$ invariant under $\Sigma \circ \tau^{-1}$. Then $\Sigma \circ \tau^{-1}$ lifts to an exact automorphism $E : \text{proj}(\mathcal{R}_C) \rightarrow \text{proj}(\mathcal{R}_C)$ and $\text{proj}(\mathcal{R}_C)/\widehat{E}$ is a 2-Calabi-Yau realization (in the sense of [13]) of a cluster algebra with geometric coefficients of type Δ . Moreover, if $C = \mathbb{Z}Q_0$ (*i.e.* $\mathcal{R}_C = \mathcal{R}$) then $\text{proj}(\mathcal{R})/\widehat{E}$ is a 2-Calabi-Yau realization of the cluster algebra with *universal coefficients* of type Δ .

Recall that cluster algebras with universal coefficients were introduced by Fomin and Zelevinsky in [11] and further investigated by Reading in [25, 26, 27]. These are cluster algebras which are universal (with respect to coefficient specialization) among cluster algebras associated to a fixed quiver. The existence of a universal coefficients for finite-type cluster algebras was carried out in [11]. In [25], it was shown that universal coefficients always exist if we restrict to the class of geometric cluster algebras and allow them to have an infinite number of coefficients (whose powers can be taken not only in the ring of integers numbers, but in more general rings such as the rational or real numbers).

Part of the technical aspects of our construction were carried out in [21], where we studied completed orbit categories associated to Frobenius categories in a more general framework. It is worth pointing out that we consider completed orbit categories rather than usual orbit categories because the Krull-Schmidt property in general is lost when passing to the orbit category, whereas it is always preserved for completed orbit categories.

We can use the insight of [20] to classify the Frobenius models of \mathcal{C}_Q satisfying conditions (i) and (ii) using completed orbit categories associated to categories of the form $\text{proj}(\mathcal{R}_C)$. Notice that we shall modify slightly the definition in [20] and admit Frobenius models to be **Hom**-infinite since the categories $\text{proj}(\mathcal{R}_C)/\widehat{E}$ have in general infinite-dimensional morphism spaces. Moreover, we can classify the Frobenius models not only of cluster categories but of larger class of orbit categories associated to $\mathcal{D}^b(\text{mod}(kQ))$. This classification problem was already addressed by Scherotzke in [31, Section 5] using usual orbit categories. The following theorem proved in this note extends Theorem 5.7 of [31].

Theorem. Let $F : \mathcal{D}^b(\text{mod}(kQ)) \rightarrow \mathcal{D}^b(\text{mod}(kQ))$ be a triangle equivalence such that $\mathcal{D}^b(\text{mod}(kQ))/F$ is **Hom**-finite and equivalent to its triangulated hull (in the sense of [19]). Let C be an admissible configuration invariant under F . Suppose moreover that for each indecomposable object X of $\text{proj}(\mathcal{R}_C)$ the group $\text{proj}(\mathcal{R}_C)(X, F_*^l(X))$ vanishes for all $l < 0$. Then

- (a) Then completed orbit category $\mathbf{proj}(\mathcal{R}_C)/F_*$ admits the structure of a Frobenius category whose stable category is triangle equivalent to \mathcal{D}_Q/F .
- (b) The category $\mathbf{proj}(\mathcal{R}_C)/F_*$ satisfies conditions (i) and (ii) above.
- (c) The map taking C to $\mathbf{proj}(\mathcal{R}_C)$ induces a bijection from the set of F -invariant admissible configurations $C \subset \mathbb{Z}Q_0$ onto the set of equivalence classes of Frobenius models $\mathbf{proj}(\mathcal{R}_C)$ of \mathcal{D}_Q/F satisfying (i) and (ii).

The main difference between [31, Theorem 5.7] and the theorem above is that, by using completed orbit categories, we are able to consider functors $F : \mathcal{D}^b(\mathbf{mod}(kQ)) \rightarrow \mathcal{D}^b(\mathbf{mod}(kQ))$ and admissible configurations $C \subset \mathbb{Z}Q_0$ for which the usual orbit category $\mathbf{proj}(\mathcal{R}_C)/F_*$ is Hom-infinite or fails to be Krull-Schmidt (in the case F_* exists).

This paper is organized as follows. In Section 2 we recall the definition and some of the fundamental properties of Nakajima categories. In Section 3 we state our main results and prove them in Sections 3 and 4. People familiarized with the cluster categories are invited to read the example first.

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2. RECOLLECTIONS

In this note we will freely use the basic concepts in the theory of dg categories. Some of the standard references for this topic are [18] and [9]. Most of the results on dg categories and dg orbit categories that will be used in this note are presented as well in Section 2 of [21]. Here we use the same notation as the one used in [21].

Notation 1. The set of morphisms between two objects x and y of a category \mathcal{A} is denoted by $\mathcal{A}(x, y)$. If k is a field and \mathcal{A} an additive k -category, a right \mathcal{A} -module is by definition a k -linear functor $M : \mathcal{A}^{\mathrm{op}} \rightarrow \mathbf{Mod}(k)$, where $\mathbf{Mod}(k)$ is the category of k -vector spaces. We denote by $\mathbf{Mod}(\mathcal{A})$ the category of all right \mathcal{A} -modules. Let $\mathbf{mod}(\mathcal{A})$ be the subcategory $\mathbf{Mod}(\mathcal{A})$ formed by the finitely presented modules. We let $\mathbf{proj}(\mathcal{A})$ be the full subcategory of $\mathbf{mod}(\mathcal{A})$ formed by the finitely-generated projective \mathcal{A} -modules. In particular, for each object x of \mathcal{A} , we have the finitely-generated projective \mathcal{A} -module

$$x^\wedge := \mathcal{A}(?, x) : \mathcal{A}^{\mathrm{op}} \rightarrow \mathbf{Mod}(k).$$

If the endomorphism ring of each object x of \mathcal{A} is local, each projective module over \mathcal{A} is a direct sum of modules of the form x^\wedge . The morphism space between two \mathcal{A} -modules L and M is denoted by $\mathbf{Hom}_{\mathcal{A}}(L, M)$ or simply $\mathbf{Hom}(L, M)$ when there is no risk of confusion. Finally, if $\mathcal{A} \rightarrow \mathcal{B}$ is a linear functor between essentially small additive k -categories we denote by $\mathrm{res} : \mathbf{Mod}(\mathcal{B}) \rightarrow \mathbf{Mod}(\mathcal{A})$ the restriction functor and by $\pi : \mathbf{Mod}(\mathcal{A}) \rightarrow \mathbf{Mod}(\mathcal{B})$ its left adjoint.

2.1. Nakajima categories. In this section we recall the construction of the Nakajima categories and some of their properties.

Let Q be a quiver. Let Q_0 be its set of vertices and Q_1 be its set of arrows. We suppose that Q is finite (both Q_0 and Q_1 are finite sets) and acyclic (Q has no oriented cycles). The *repetition quiver* (cf. [29]) $\mathbb{Z}Q$ is the quiver obtained from Q as follows:

- the set of vertices of $\mathbb{Z}Q$ is $\mathbb{Z}Q_0 = Q_0 \times \mathbb{Z}$.

- For each arrow $\alpha : i \longrightarrow j$ of Q and each $p \in \mathbb{Z}$, the repetition quiver $\mathbb{Z}Q$ has the arrows

$$(\alpha, p) : (i, p) \longrightarrow (j, p) \quad \text{and} \quad \sigma(\alpha, p) : (j, p-1) \longrightarrow (i, p).$$

- $\mathbb{Z}Q$ has no more arrows than the ones described above.

Let $\sigma : \mathbb{Z}Q_1 \rightarrow \mathbb{Z}Q_1$ be the bijection given by

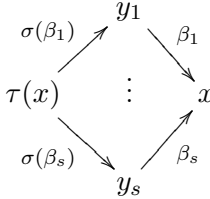
$$\sigma(\beta) = \begin{cases} \sigma(\alpha, p) & \text{if } \beta = (\alpha, p), \\ (\alpha, p-1) & \text{if } \beta = \sigma(\alpha, p). \end{cases}$$

Let $\tau : \mathbb{Z}Q \rightarrow \mathbb{Z}Q$ be the graph automorphism given by *the translation by one unit*:

$$\tau(i, p) = (i, p-1) \quad \text{and} \quad \tau(\beta) = \sigma^2(\beta)$$

for each vertex (i, p) and each arrow β of $\mathbb{Z}Q$.

Let k be a field. Following [14] and [28], we define the *mesh category* $k(\mathbb{Z}Q)$ to be the quotient of the path category $k\mathbb{Z}Q$ by the ideal generated by the mesh relators, *i.e.* the k -category whose objects are the vertices of $\mathbb{Z}Q$ and whose morphism space from a to b is the space of all k -linear combinations of paths from a to b modulo the subspace spanned by all elements ur_xv , where u and v are paths and

$$r_x = \sum_{\beta: y \rightarrow x} \beta \sigma(\beta) :$$


is the *mesh relator* associated with a vertex x of $\mathbb{Z}Q$. Here the sum runs over all arrows $\beta : y \rightarrow x$ of $\mathbb{Z}Q$.

Notation 2. Let kQ be the path algebra of Q and let $\mathbf{mod}(kQ)$ be the category of all finite-dimensional right kQ -modules. To be consistent with the notation in [20], we let \mathcal{D}_Q denote the bounded derived category $\mathcal{D}^b(\mathbf{mod}(kQ))$. The symbol $\mathbf{ind}(\mathcal{D}_Q)$ denotes the full subcategory of \mathcal{D}_Q formed by its indecomposable objects. More generally, for an additive category \mathcal{X} we write $\mathbf{ind}(\mathcal{X})$ to denote its full subcategory of indecomposable objects.

Theorem 3 ([16]). *There is a canonical fully faithful functor*

$$H : k(\mathbb{Z}Q) \rightarrow \mathbf{ind}(\mathcal{D}_Q)$$

taking each vertex $(i, 0)$ to the indecomposable projective module associated to the vertex $i \in Q_0$. It is an equivalence if and only if Q is a Dynkin quiver (= an orientation of a Dynkin diagram of type ADE).

Definition 4. ([20]) The *framed quiver* \tilde{Q} associated to Q is the quiver obtained from Q by adding, for each vertex $i \in Q_0$, a new vertex i' and a new arrow $i \rightarrow i'$. We consider the repetition quiver $\mathbb{Z}\tilde{Q}$ and call *frozen vertices* its vertices of the form (i', n) , with $i \in Q_0$ and $n \in \mathbb{Z}$. The *regular Nakajima category* \mathcal{R} associated to Q is the quotient of the path category $k\mathbb{Z}\tilde{Q}$ by the ideal generated by the mesh relators associated to the *non-frozen vertices*. The *singular Nakajima category* \mathcal{S} is the full subcategory of \mathcal{R} whose objects are the frozen vertices.

As shown in [16], via the embedding H , the autoequivalence τ of the mesh category corresponds to the *Auslander–Reiten translation* of \mathcal{D}_Q , which we will also denote by τ . For Dynkin quivers, the combinatorial descriptions of ν , Σ and of the image of $\mathbf{mod}(kQ)$

in \mathcal{D}_Q are given in section 6.5 of [14]. If Q is Dynkin, let Σ be the unique bijection of the vertices of $\mathbb{Z}Q$ such that

$$H(\Sigma x) = \Sigma(H(x)).$$

Remark 5. Notice that $\Sigma : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}Q_0$ extends to a bijection $\Sigma : \mathbb{Z}\tilde{Q}_0 \rightarrow \mathbb{Z}\tilde{Q}_0$. There is also a bijection $\sigma : \mathbb{Z}\tilde{Q}_0 \rightarrow \mathbb{Z}\tilde{Q}_0$ given by $\sigma : (i, n) \mapsto (i', n - 1)$ and $(i', n) \mapsto (i, n)$ for i a vertex of Q and n an integer.

Example 6. Let Q be the Dynkin quiver $1 \rightarrow 2$. The quiver $\mathbb{Z}\tilde{Q}$ is depicted in Figure 1 below. The frozen vertices are represented by small squares \square . In this case the mesh relations imply that $ba + dc = 0$ in the Nakajima category whereas $eb \neq 0$.

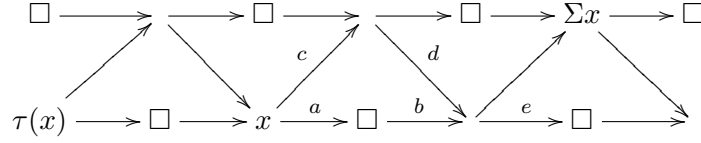


FIGURE 1. The quiver of the regular Nakajima category associated to A_2 .

Definition 7. Let C be a subset of $\mathbb{Z}Q_0$. Denote by \mathcal{R}_C the quotient of \mathcal{R} by the ideal generated by the identities of the frozen vertices not belonging to $\sigma^{-1}(C)$ and by \mathcal{S}_C its full subcategory formed by the vertices in $\sigma^{-1}(C)$. We call C an *admissible configuration* of $\mathbb{Z}Q$ if for each vertex x of $\mathbb{Z}Q$, there is a vertex c in C such that the space of morphisms from x to c in the mesh category $k(\mathbb{Z}Q)$ does not vanish. Finally, let $\mathbb{Z}Q_C$ be the quiver obtained from $\mathbb{Z}\tilde{Q}$ by delating the set of frozen vertices $\sigma^{-1}(C)$.

2.2. Nakajima categories and derived categories. Consider the category of finitely presented \mathcal{S}_C -modules $\text{mod}(\mathcal{S}_C)$.

Definition 8. An \mathcal{S}_C -module M is *finitely generated Gorenstein projective* if there is an acyclic complex

$$P_M : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

of objects in $\text{proj}(\mathcal{S}_C)$ such that $M \cong \text{cok}(P_1 \rightarrow P_0)$ and the complex $\text{Hom}_{\mathcal{S}_C}(P_M, P')$ is still acyclic for each module P' in $\text{proj}(\mathcal{S}_C)$. Denote by $\text{gpr}(\mathcal{S}_C)$ the full subcategory of $\text{mod}(\mathcal{S}_C)$ formed by the Gorenstein projective modules. In the situation described above we call P_M a *complete projective resolution* of M .

Notice that every finitely generated projective \mathcal{A} -module P lies in $\text{gpr}(\mathcal{A})$, since we may take its complete projective resolution as $\cdots \rightarrow 0 \rightarrow P \xrightarrow{\sim} P \rightarrow 0 \rightarrow \cdots$. The following is a well-known result.

Lemma 9. *The category $\text{gpr}(\mathcal{S}_C)$ is an extension closed subcategory of $\text{Mod}(\mathcal{S}_C)$. Moreover, the induce exact structure on $\text{gpr}(\mathcal{S}_C)$ makes it a Frobenius category whose subcategory of projective-injective objects is $\text{proj}(\mathcal{S}_C)$.*

Remark 10. The Gorenstein injective modules are defined analogously, all the results in this note can be naturally adapted to be stated in terms of Gorenstein injective modules.

The following theorem summarizes some the results of [20] that we shall need.

Theorem 11. ([20]) *Let Q be a Dynkin quiver and C an admissible configuration of $\mathbb{Z}Q$. Then*

(i) the restriction functor

$$\text{res} : \text{Mod}(\mathcal{R}_C) \rightarrow \text{Mod}(\mathcal{S}_C)$$

induces an equivalence between the full subcategory of finitely generated projective \mathcal{R}_C -modules $\text{proj}(\mathcal{R}_C)$ and the category $\text{gpr}(\mathcal{S}_C)$

$$\text{res} : \text{proj}(\mathcal{R}_C) \xrightarrow{\sim} \text{gpr}(\mathcal{S}_C).$$

In particular, it yields an isomorphism of $\mathbb{Z}\tilde{Q}_C$ onto the Auslander-Reiten quiver of $\text{gpr}(\mathcal{S}_C)$ so that the vertices of $\sigma^{-1}(C)$ correspond to the projective-injective objects,
(ii) there is a δ -functor $\Phi : \text{mod}(\mathcal{S}_C) \rightarrow \mathcal{D}_Q$. It can be factorized as

$$\text{gpr}(\mathcal{S}_C) \xrightarrow{\Omega} \underline{\text{gpr}}(\mathcal{S}_C) \xrightarrow{\phi} \mathcal{D}_Q,$$

where Ω is the syzygy functor and ϕ is a triangle equivalence.

Remark 12. In view of part (i) of Theorem 11, the indecomposable objects of $\text{gpr}(\mathcal{S}_C)$ are of the form $\text{res}(x^\wedge)$, for a vertex x of $\mathbb{Z}Q_C$. If y is a frozen vertex of $\mathbb{Z}Q_C$, then the projective \mathcal{S}_C -module $y^\wedge \in \text{gpr}(\mathcal{S}_C)$ is identified with $\text{res}(y^\wedge)$. For simplicity, we denote the indecomposable objects of \mathcal{E} by x^\wedge , for some vertex x of $\mathbb{Z}Q_C$.

Assumption 13. From now on we suppose that Q is an orientation of a connected and simply laced Dynkin diagram.

2.3. Standard Frobenius models of \mathcal{D}_Q . Let k be an algebraically closed field and \mathcal{T} a triangulated k -linear category. A *Frobenius model* of \mathcal{T} is a Frobenius category \mathcal{E} together with a triangle equivalence $\mathcal{T} \cong \underline{\mathcal{E}}$. From Theorem 11 we know that if $C \subset \mathbb{Z}Q_0$ is an admissible configuration, then the category

$$\mathcal{E}_C := \text{gpr}(\mathcal{S}_C) \cong \text{proj}(\mathcal{R}_C)$$

is a Frobenius model of \mathcal{D}_Q . The Frobenius models obtained in this way were characterized in [20]. They are exactly the Frobenius categories satisfying conditions (P0)-(P3) bellow.

For an arbitrary Frobenius category \mathcal{E} , consider the following properties:

- (P0) The category \mathcal{E} is k -linear, Ext-finite and Krull-Schmidt.
- (P1) For each indecomposable non projective object X of \mathcal{E} , there is an almost split sequence starting and an almost split sequence ending at X .
- (P2) For each indecomposable projective object P of \mathcal{E} , the \mathcal{E} -module $\text{rad}_{\mathcal{E}}(?, P)$ and the \mathcal{E}^{op} -module $\text{rad}_{\mathcal{E}}(P, ?)$ are finitely generated with simple tops.
- (P3) \mathcal{E} is standard, i.e. its category of indecomposables is equivalent to the mesh category of its Auslander-Reiten quiver (cf. section 2.3, page 63 of [30]).

Theorem 14. ([20, Corollary 5.25]) *The map taking C to \mathcal{E}_C induces a bijection from the set of admissible configurations $C \subset \mathbb{Z}Q_0$ onto the set of equivalence classes of Frobenius models \mathcal{E} of \mathcal{D}_Q satisfying (P0) – (P3). The inverse bijection sends a Frobenius model \mathcal{E} to the set $C \subset \mathbb{Z}Q_0$ such that the indecomposable projectives of \mathcal{E} correspond to the vertices $\sigma^{-1}(c)$, $c \in C$, of the Auslander-Reiten quiver of \mathcal{E} .*

Remark 15. The existence of almost split triangles in the stable category $\underline{\mathcal{E}}$ implies condition (P1) so that this condition holds in particular in all Frobenius models of \mathcal{D}_Q .

3. THE MAIN RESULTS

This section is devoted to state our main results. Their proofs are given in the subsequent sections.

3.1. Completed orbit categories. Let \mathcal{A} be a k -linear category and $F : \mathcal{A} \rightarrow \mathcal{A}$ an automorphism. By definition, the orbit category \mathcal{A}/F has the same objects as \mathcal{A} , the set of morphisms from an object X to an object Y is given by

$$\mathcal{A}/F(X, Y) = \bigoplus_{l \in \mathbb{Z}} \mathcal{A}(X, F^l(Y)).$$

The composition of morphisms is given by the formula

$$(3.1) \quad (f_a) \circ (g_b) = \left(\sum_{a+b=c} F^b(f_a) \circ g_b \right),$$

where $f_a : Y \rightarrow F^a(Z)$, $g_b : X \rightarrow F^b(Y)$ and $a, b \in \mathbb{Z}$. Clearly \mathcal{A}/F is still a k -linear category and the canonical projection $p : \mathcal{A} \rightarrow \mathcal{A}/F$ is an additive functor. Suppose that for all objects X, Y of \mathcal{A} , the space $\mathcal{A}(X, F^l(Y))$ vanishes for all integers $l \ll 0$. In this case we can define the *completed orbit category* $\widehat{\mathcal{A}/F}$ as the category whose objects are the same as those of \mathcal{A} and with morphism spaces

$$(3.2) \quad \widehat{\mathcal{A}/F}(X, Y) = \prod_{l \in \mathbb{Z}} \mathcal{A}(X, F^l(Y)).$$

Notice that the vanishing condition imposed on the spaces $\mathcal{A}(X, F^l(Y))$ ensures that the composition in $\widehat{\mathcal{A}/F}$ defined as for the usual orbit category in (3.1) is a well-defined operation. Clearly, the category $\widehat{\mathcal{A}/F}$ is k -linear. We still denote the natural projection $\mathcal{A} \rightarrow \widehat{\mathcal{A}/F}$ by p .

Assumption 16. Whenever we make reference to the completed orbit category associated to an automorphism $F : \mathcal{A} \rightarrow \mathcal{A}$ we will implicitly assume that it is defined. Namely, we assume that for all objects X, Y of \mathcal{A} , the space $\mathcal{A}(X, F^l(Y))$ vanishes for all integers $l \ll 0$.

Remark 17. A standard construction allows one to replace a category with autoequivalence by a category with an automorphism (see Section 7 of [1]). So in what follows we may consider orbit categories associated to categories equipped with an automorphism.

Remark 18. Each indecomposable object of $\widehat{\mathcal{A}/F}$ (and $\widehat{\mathcal{A}/F}_*$) is the image of an indecomposable object of \mathcal{A} under p .

Now if \mathcal{B} is a dg category endowed with an endomorphism $F : \mathcal{B} \rightarrow \mathcal{B}$ inducing an equivalence $H^0(F) : H^0(\mathcal{B}) \rightarrow H^0(\mathcal{B})$, then we define the *dg orbit category* \mathcal{B}/F as follows: the objects of \mathcal{B}/F are the same as the objects of \mathcal{B} . For $X, Y \in \mathcal{B}/F$, we have

$$(3.3) \quad \mathcal{B}/F(X, Y) := \operatorname{colim}_p \bigoplus_{n \geq 0} \mathcal{B}(F^n(X), F^n(Y)),$$

where the transitions maps are given by F

$$\bigoplus_{n \geq 0} \mathcal{B}(F^n(X), F^n(Y)) \xrightarrow{F} \bigoplus_{n \geq 0} \mathcal{B}(F^n(X), F^{n+1}(Y)).$$

Definition 19. Let \mathcal{T} be a triangulated category endowed with a triangulated equivalence $F : \mathcal{T} \rightarrow \mathcal{T}$. Suppose that \mathcal{T}_{dg} is a dg enhancement of \mathcal{T} and that $\tilde{F} : \mathcal{T}_{dg} \rightarrow \mathcal{T}_{dg}$ a dg functor such that $H^0(\tilde{F}) = F$. The triangulated hull of \mathcal{T}/F (with respect to \tilde{F}) is the triangulated category $H^0(\operatorname{pretr}(\mathcal{T}_{dg}/\tilde{F}))$, where $\operatorname{pretr}(\mathcal{T}_{dg}/\tilde{F})$ is the pretriangulated hull of $\mathcal{T}_{dg}/\tilde{F}$.

3.2. Frobenius models of \mathcal{D}_Q/F and categorification of cluster algebras. Let $F : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$ be a triangle equivalence which is isomorphic to the derived tensor product

$$? \otimes_{kQ}^L M : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$$

for some complex M of kQ -bimodules. Notice that this is not a restriction since all autoequivalences with an "algebraic" construction are of this form (cf. Section 9 of [19]).

Let $C \subset \mathbb{Z}Q$ be an admissible configuration which is F -invariant, i.e. $F(C) = C$. Then F induces an automorphism on \mathcal{R}_C which we still denote by F . Let F_* be the automorphism on $\text{Mod}(\mathcal{R}_C)$ induced by F :

$$F_*(M) = M \circ F^{-1}.$$

In particular, $F_*(x^\wedge) = (F(x))^\wedge$ for every vertex x of $\mathbb{Z}Q_C$, so F_* restricts to an exact automorphism $F_* : \mathcal{E}_C \rightarrow \mathcal{E}_C$.

Consider the dg functor

$$(3.4) \quad \tilde{F} := -? \otimes_{kQ} P^* M : C^b(\text{proj}(kQ))_{dg} \rightarrow C^b(\text{proj}(kQ))_{dg}$$

where $P^* M$ is a projective resolution of the bimodule M . We have that $H^0(\tilde{F}) = F$.

Theorem 20. *Let $F : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$ be a triangle equivalence such that \mathcal{D}_Q/F is **Hom**-finite and equivalent to its triangulated hull. Let C be an admissible configuration invariant under F . Suppose moreover that for each indecomposable object X of \mathcal{E}_C the group $\mathcal{E}_C(X, F_*^l(X))$ vanishes for all $l < 0$. Then*

- (i) *Then completed orbit category $\mathcal{E}_C/\widehat{F}_*$ admits the structure of a Frobenius category whose stable category is triangle equivalent to \mathcal{D}_Q/F (see Assumption 16).*
- (ii) *The category $\mathcal{E}_C/\widehat{F}_*$ satisfies conditions $(P_0) - (P_3)$ of Section 2.3 and its AR quiver is isomorphic to $\mathbb{Z}\tilde{Q}_C/F$.*
- (iii) *The map taking C to \mathcal{E}_C induces a bijection from the set of F -invariant admissible configurations $C \subset \mathbb{Z}Q_0$ onto the set of equivalence classes of Frobenius models \mathcal{E} of \mathcal{D}_Q/F satisfying $(P_0) - (P_3)$.*

Notice, that in the situation of Theorem 20, the completed orbit category $\mathcal{E}_C/\widehat{F}_*$ is in general **Hom**-infinite, but it is always **Ext**¹-finite since \mathcal{D}_Q/F is **Hom**-finite. From [19] we know that \mathcal{D}_Q/F is **Hom**-finite and equivalent to its triangulated hull for example if the following conditions hold:

- (a) *for each indecomposable U of $\text{mod } kQ$, there are only finitely many $i \in \mathbb{Z}$ such that the object $F^i(U)$ lies in $\text{mod } kQ$,*
- (b) *there is an integer $N \geq 0$ such that the F -orbit of each indecomposable of \mathcal{D}_Q^b contains an object $\Sigma^n U$, for some $0 \leq n \leq N$ and some indecomposable object U of $\text{mod } kQ$,*

Theorem 21. *Let Δ be the simply laced Dynkin diagram which underlies the quiver Q . Let F be the autoequivalence $\Sigma \circ \tau^{-1} : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$ and $C \subset \mathbb{Z}Q$ be an admissible configuration invariant under F .*

- (i) *Then $\mathcal{E}_C/\widehat{F}_*$ is a 2-Calabi-Yau realization (in the sense of [13]) of a cluster algebra with geometric coefficients of type Δ .*
- (ii) *If $C = \mathbb{Z}Q$ then $\mathcal{E}_{\mathbb{Z}Q}/\widehat{F}_*$ is a 2-Calabi-Yau realization of the cluster algebra with universal coefficients of type Δ (cf. [11]).*

The rest of the article is devoted to prove these theorems.

4. FROBENIUS MODELS OF \mathcal{D}_Q/F

In this section we prove Theorem 20.

4.1. The stratifying functor. The functor $\Phi : \text{mod}(\mathcal{S}_C) \rightarrow \mathcal{D}_Q$ of Theorem 11 was named in [20] the *stratifying functor*. It can be used to obtain a stratification of Nakajima's graded affine quiver variety (introduced in [22]) whose strata are parameterized by $\text{ind}(\mathcal{D}_Q)$. We will show that $\phi : \underline{\text{gpr}}(\mathcal{S}_C) \rightarrow \mathcal{D}_Q$ is induced by a quasi-functor. The inverse of $\phi : \underline{\text{gpr}}(\mathcal{S}_C) \rightarrow \mathcal{D}_Q$ can be described explicitly as the composition of two functors as follows: On the one hand we consider the path category kQ as a full subcategory of \mathcal{R}_C via the embedding $i \mapsto (i, 0)$. The restriction functor gives a functor $kQ \rightarrow \mathcal{S}_C$ taking x to $\text{res}(x^\wedge)$. It gives rise to a kQ - \mathcal{S}_C -bimodule X given by

$$X(u, x) = \text{Hom}(u^\wedge, \text{res}(x^\wedge)), \text{ for } x \in Q_0 \text{ and } u \in \sigma(C)$$

and therefore a functor

$$? \otimes_{kQ}^L X : \mathcal{D}_Q \rightarrow \mathcal{D}^b(\text{mod}(\mathcal{S}_C)).$$

Notice that for every kQ -module M the \mathcal{S}_C -module $M \otimes_{kQ} X$ lies in $\text{gpr}(\mathcal{S}_C)$. By definition, the derived category of $\text{gpr}(\mathcal{S}_C)$ (as an exact category) can be identified with a full triangulated subcategory of $\mathcal{D}^b(\text{mod}(\mathcal{S}_C))$. Therefore we can consider the derived tensor product as a triangulated functor

$$? \otimes_{kQ}^L X : \mathcal{D}_Q \rightarrow \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)).$$

On the other hand, since $\text{gpr}(\mathcal{S}_C)$ is a Frobenius category, there is a canonical triangulated functor

$$\text{can} : \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \rightarrow \underline{\text{gpr}}(\mathcal{S}_C).$$

Let us recall the construction of can . Let \mathcal{P} be the full subcategory of $\text{gpr}(\mathcal{S}_C)$ formed by its projective objects and denote by $\mathcal{H}^-(\mathcal{P})$ the homotopy category associated to the category of bounded above complexes with components in \mathcal{P} . There is a functor

$$\mathbf{p} : \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \rightarrow \mathcal{H}^-(\mathcal{P})$$

sending a complex X to a quasi-isomorphic complex $\mathbf{p}X \in \mathcal{H}^-(\mathcal{P})$. If P^\cdot is a complex in $\mathcal{H}^-(\mathcal{P})$ and $p \in \mathbb{Z}$ is small enough, then the objects $\Sigma^{-p}(Z^p(P^\cdot)) \in \underline{\text{gpr}}(\mathcal{S}_C)$ are canonically isomorphic. Moreover, if P_1^\cdot and P_2^\cdot are quasi-isomorphic complexes in $\mathcal{H}^-(\mathcal{P})$ and $p \ll 0$ then we have an isomorphism $\Sigma^{-p}(Z^p(P_1^\cdot)) \cong \Sigma^{-p}(Z^p(P_2^\cdot))$ in $\underline{\text{gpr}}(\mathcal{S}_C)$. Thus there is a functor

$$t : \mathcal{H}^-(\mathcal{P}) \rightarrow \underline{\text{gpr}}(\mathcal{S}_C)$$

sending a complex P^\cdot to $t(P^\cdot) := \Sigma^{-p}(Z^p(P^\cdot))$ for some $p \ll 0$ (which depends on P^\cdot). Then $\text{can} : \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \rightarrow \underline{\text{gpr}}(\mathcal{S}_C)$ is defined as the composition of the functors described above

$$\text{can} = t \circ \mathbf{p}.$$

Finally, the functor $\phi^{-1} : \mathcal{D}_Q \rightarrow \underline{\text{gpr}}(\mathcal{S}_C)$ is given by the composition

$$\phi^{-1} : \mathcal{D}_Q \xrightarrow{? \otimes_{kQ}^L X} \mathcal{D}^b(\text{gpr}(\mathcal{S}_C)) \xrightarrow{\text{can}} \underline{\text{gpr}}(\mathcal{S}_C).$$

Remark 22. The fact that ϕ^{-1} is a triangle equivalence is proved in Section 5 of [20].

4.2. A dg lift of ϕ^{-1} . Let us prove that the functor ϕ^{-1} admits a dg lift the sense of [19]. In other words, we prove that ϕ^{-1} is the triangulated functor associated to a quasi-functor between pretriangulated dg categories.

Notation 23. Let $C \subset \mathbb{Z}Q_0$ be an admissible configuration. Recall that we have defined \mathcal{E}_C to be the Frobenius category $\text{gpr}(\mathcal{S}_C)$ (or equivalently the category $\text{proj}(\mathcal{R}_C)$). In the rest of the section we will drop the subindex C and write for simplicity just \mathcal{E} . We let \mathcal{P} be the full subcategory \mathcal{E} formed by its projective objects.

Let $C^b(\text{proj}(kQ))_{dg}$ be the dg category of bounded complexes of projective kQ -modules and $\mathcal{A}c(\mathcal{P})_{dg}$ be the dg category of acyclic complexes of objects of \mathcal{P} . The dg categories $C^b(\text{proj}(kQ))_{dg}$ and $\mathcal{A}c(\mathcal{P})_{dg}$ are canonical dg enhancements of the categories \mathcal{D}_Q and $\underline{\mathcal{E}}$, respectively. The triangulated category $\mathcal{D}^b(\mathcal{E})$ admits as canonical dg enhancement the dg quotient

$$\mathcal{D}^b(\mathcal{E})_{dg} := C^b(\mathcal{E})_{dg} / \mathcal{A}c^b(\mathcal{E})_{dg}.$$

We will show that the equivalence ϕ^{-1} is the triangulated functor associated to a quasi-functor $C^b(\text{proj}(kQ))_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$.

Definition 24. Let $\text{dgc}at_k$ denote the category of small dg categories over k . It admits the structure of a model category whose weak equivalences are the quasi-equivalences (cf. [32]). Let Hq denote the associated homotopy category. Let \mathcal{A} be a small dg category and \mathcal{B} a full dg subcategory of \mathcal{A} . We say that a morphism $G : \mathcal{A} \rightarrow \mathcal{A}'$ in Hq *annihilates* \mathcal{B} if its associated functor $H^0(G) : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{A}')$ takes all objects of \mathcal{B} to zero objects.

Theorem 25. ([18, 33]) *There is a morphism $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ of Hq which annihilates \mathcal{B} and is universal among the morphisms annihilating \mathcal{B} . Moreover, the morphism $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ induces an equivalence between the category of quasi-functors $\mathcal{A}/\mathcal{B} \rightarrow \mathcal{A}'$ into the category of quasi-functors $\mathcal{A} \rightarrow \mathcal{A}'$ whose associated functor annihilates \mathcal{B} .*

Proposition 26. *There is a quasi-functor*

$$\tilde{\phi}^{-1} : C^b(\text{proj}(kQ))_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$$

such that $H^0(\tilde{\phi}^{-1}) \cong \phi^{-1}$.

Proof. Recall that ϕ^{-1} is defined as the composition

$$\phi^{-1} : \mathcal{D}_Q \xrightarrow{? \otimes_{kQ}^L X} \mathcal{D}^b(\mathcal{E}) \xrightarrow{\text{can}} \underline{\mathcal{E}}.$$

We will show that these two functors admit a dg lift. Let $P \cdot X$ be a projective resolution of X as a kQ - \mathcal{S}_C -bimodule. We can compose the dg functor

$$-? \otimes_{kQ} P \cdot X : C^b(\text{proj}(kQ))_{dg} \rightarrow C^b(\mathcal{E})_{dg}$$

with the canonical dg functor $C^b(\mathcal{E})_{dg} \rightarrow \mathcal{D}^b(\mathcal{E})_{dg}$ to obtain a dg lift of $? \otimes_{kQ}^L X : \mathcal{D}_Q \rightarrow \mathcal{D}^b(\mathcal{E})$. Recall that every module $M \in \mathcal{E}$ is of the form $M \cong Z^0(P_M)$, for some complex $P_M \in \mathcal{A}c(\mathcal{P})$. Moreover, since projective resolutions and injective coresolutions of objects in \mathcal{E} can be chosen functorially, there is a faithful functor

$$i : \mathcal{E} \rightarrow \mathcal{A}c(\mathcal{P}).$$

Therefore, we can consider $C^b(\mathcal{E})$ as a subcategory of the category $\text{Bi}(\mathcal{P})$ of *double complexes* with components in \mathcal{P} (see for instances **Sign Trick 1.2.5** of [34]). Let $\text{Tot}(B)$ be the *completed* total complex associated to a double complex $B \in \text{Bi}(\mathcal{P})$. By construction, we have that

$$\text{can}(M) \cong H^0(\text{Tot}(i(M))).$$

The composition $\text{Tot} \circ i : C^b(\mathcal{E}) \rightarrow \mathcal{A}c(\mathcal{P})$ defines a dg functor $\text{Tot} \circ i : C^b(\mathcal{E})_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$ which annihilates $\mathcal{A}c^b(\mathcal{E})_{dg}$. In light of Theorem 25, there is a quasi-functor

$$\widehat{\text{can}} : \mathcal{D}^b(\mathcal{E})_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$$

whose associated triangle functor is $\text{can} : \mathcal{D}^b(\mathcal{E}) \rightarrow \mathcal{E}$. Finally, we can define $\tilde{\phi}^{-1}$ as the quasi-functor associated to the composition of dg functors

$$\tilde{\phi}^{-1} : C^b(\text{proj}(kQ))_{dg} \xrightarrow{? \otimes_{kQ} P^X} C^b(\mathcal{E})_{dg} \xrightarrow{\widehat{\text{can}}} \mathcal{A}c(\mathcal{P})_{dg}.$$

□

4.3. Proof of Theorem 20. Recall that we are given an exact autoequivalence $F_* : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ extending a triangle functor $F : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$ such that \mathcal{D}_Q/F is **Hom**-finite and triangulated with respect to \tilde{F} (see (3.4)). Since F_* is exact, it restricts to an equivalence $F_* : \mathcal{P} \rightarrow \mathcal{P}$ and therefore it induces a triangulated functor $\underline{F}_* : \underline{\mathcal{E}} \rightarrow \underline{\mathcal{E}}$. Let $\tilde{F}_* : \mathcal{A}c(\mathcal{P})_{dg} \rightarrow \mathcal{A}c(\mathcal{P})_{dg}$ be the dg functor defined as F_* componentwise, it induces \underline{F}_* in homology, *i.e.* $H^0(\tilde{F}_*) = \underline{F}_*$.

Lemma 27. *Under the above assumptions, the dg category $\mathcal{A}c(\mathcal{P})_{dg}/\tilde{F}_*$ is quasi-equivalent to $C^b(\text{proj } kQ)_{dg}/\tilde{F}$. In particular, the triangulated hull of $\underline{\mathcal{E}}/\underline{F}_*$ is triangle equivalent to the triangulated hull of \mathcal{D}_Q/F .*

Proof. Let $\tilde{\phi}$ be an inverse of the quasi-functor $\tilde{\phi}^{-1}$. In view of the universal property of the triangulated hull (see Section 9.4 of [19]) it is enough to show that the diagram

$$\begin{array}{ccc} \mathcal{A}c(\mathcal{P})_{dg} & \xrightarrow{M_{\tilde{\phi}}} & C^b(\text{proj } kQ)_{dg} \\ M_{\tilde{F}_*} \downarrow & & \downarrow M_{\tilde{F}} \\ \mathcal{A}c(\mathcal{P})_{dg} & \xrightarrow{M_{\tilde{\phi}}} & C^b(\text{proj } kQ)_{dg}. \end{array}$$

is commutative up to isomorphism, where M_X is the quasi functor associated to a dg functor X . By construction we have that $F\phi \cong \phi F_*$. Thus there is a isomorphism of quasi functors

$$M_{\tilde{F}} M_{\tilde{\phi}} \cong M_{\tilde{\phi}} M_{\tilde{F}_*}.$$

□

Proof of Theorem 20. (i) By Lemma 27 the category $\underline{\mathcal{E}}/\underline{F}_*$ is triangulated and triangle equivalent to \mathcal{D}_Q/F . By Theorem 42 of [21], $\widehat{\mathcal{E}}/\widehat{F}_*$ admits the structure of a Frobenius category whose stable category is triangle equivalent to \mathcal{D}_Q/F .

(ii) It is clear that $\widehat{\mathcal{E}}/\widehat{F}_*$ is k -linear. By Theorem 42 of [21] it is Krull-Schmidt (see also Lemma 35 of *loc. cit.*). It is Ext^1 -finite since \mathcal{D}_Q/F is **Hom**-finite. So $\widehat{\mathcal{E}}/\widehat{F}_*$ satisfies condition (P0).

We know that the canonical projection $p : \mathcal{D}_Q \rightarrow \mathcal{D}_Q/F$ is exact. Then by the proof of Proposition 1.3 of [4] we have that \mathcal{D}_Q/F satisfies condition (P1) and that its AR quiver is $\mathbb{Z}Q/F$. Therefore, $\widehat{\mathcal{E}}/\widehat{F}_*$ satisfies (P1) as well.

Let $p(z^\wedge)$ be an indecomposable projective object of $\widehat{\mathcal{E}}/\widehat{F}_*$ (See Remark 12 and Remark 18). So z is some frozen vertex of $\mathbb{Z}Q_C$. Let f be the morphism in \mathcal{E} corresponding to be the arrow $z \rightarrow \sigma^{-1}(z)$ in $\mathbb{Z}Q_C$ under the Yoneda embedding $\mathcal{R}_C \rightarrow \mathcal{E}$. By Theorem 14, f is an irreducible morphism. We claim that the image of f under the canonical projection $p : \mathcal{E} \rightarrow \widehat{\mathcal{E}}/\widehat{F}_*$ is an irreducible morphism. Let $p(x^\wedge) \in \widehat{\mathcal{E}}/\widehat{F}_*$ be an indecomposable object which is not isomorphic to $p(z^\wedge)$. Let $h : p(z^\wedge) \rightarrow p(x^\wedge)$ be a non-zero morphism in $\widehat{\mathcal{E}}/\widehat{F}_*$.

Then $h = (h_i)$, with $h_i : z^\wedge \rightarrow F_*^i(x^\wedge)$. Since $p(z^\wedge)$ is not isomorphic to $p(x^\wedge)$, we have that $F^i(x) \neq z$ for all $i \in Z$. Since f is irreducible, we have that $h_i = g_i \circ f$ for some $g_i : \sigma^{-1}(z)^\wedge \rightarrow F_*^i(x^\wedge)$. Then $h = (g_i) \circ p(f)$. This shows that $p(f)$ is left almost split and that $p(f)$ is the only irreducible morphism with source $p(z^\wedge)$. Similarly, let g be the irreducible morphism in \mathcal{E} corresponding to the arrow $\sigma(z) \rightarrow z$ in $\mathbb{Z}Q_C$. As before we conclude that $p(g)$ the only irreducible morphism of $\mathcal{E}/\widehat{F}_*$ ending on $\sigma(z)^\wedge$. This shows that the map $p(x^\wedge) \rightarrow x$ induces an isomorphism between the AR quiver of $\mathcal{E}/\widehat{F}_*$ and the quiver $\mathbb{Z}Q_C/F$. Moreover, the frozen vertices of $\mathbb{Z}Q_C$ correspond to the projective-injective objects of $\mathcal{E}/\widehat{F}_*$.

It only remains to show that $\mathcal{E}/\widehat{F}_*$ is standard. We have to check that there are no relations between frozen vertices of $\mathbb{Z}Q_C/F$. Consider the functor $\pi : \mathbf{Mod}(\mathcal{E}) \rightarrow \mathbf{Mod}(\mathcal{E}/\widehat{F}_*)$, *i.e.* the left adjoint of the restriction functor $\text{res} : \mathbf{Mod}(\mathcal{E}) \rightarrow \mathbf{Mod}(\mathcal{E}/\widehat{F}_*)$ associated to the canonical projection $\mathcal{E} \rightarrow \mathcal{E}/\widehat{F}_*$. By Lemma 34 of [21], we know that π preserve simple modules. Let $x \neq y$ be two frozen vertices of $\mathbb{Z}Q_C$. We know from [3] that the number of minimal relations starting at a vertex x and ending at the y equals the dimension of $\text{Ext}_{\mathcal{E}/\widehat{F}_*}^2(\pi(S_y), \pi(S_x))$, where S_x (resp. S_y) is the simple \mathcal{E} -module concentrated in $p(x^\wedge)$ (resp. $p(y^\wedge)$). By Lemma 34 of [21] we have that

$$\text{Ext}_{\mathcal{E}/\widehat{F}_*}^2(\pi(S_x), \pi(S_y)) = \prod_{i \in \mathbb{Z}} \text{Ext}_{\mathcal{E}}^2(S_x, F^l(S_y)).$$

Since \mathcal{E} is standard, we have that $\text{Ext}_{\mathcal{E}}^2(S_x, F^l(S_y)) = 0$ for all $l \in \mathbb{Z}$. The claim follows.

(iii) Now we only have to check that C is admissible. Let $\pi(x^\wedge) \xrightarrow{(f_i)} \pi(I)$ be an inflation, where I is an injective object of \mathcal{E} and $f_i : x^\wedge \rightarrow F^i(I)$ is a morphism of \mathcal{E} . In particular, there is a path p from x to $\sigma^{-1}(c)$ for some c in C . To finish the proof we can proceed exactly as in the proof of Theorem 14 given [20]. \square

5. CATEGORIFICATION

This section is devoted to prove Theorem 21. For the rest of the section we assume that the functor F corresponds to $\Sigma \circ \tau^{-1}$.

5.1. 2-CY realization. In this subsection we prove part (i) of Theorem 21. So let us first recall some essential concepts. Let \mathcal{F} be a Frobenius category.

The category \mathcal{F} is stably 2-Calabi-Yau (2-CY for short) if its stable category $\underline{\mathcal{F}}$ is an Hom-finite triangulated category which is 2-CY, *i.e.* for every pair of objects X and Y of $\underline{\mathcal{F}}$ there is an isomorphism

$$\text{Hom}_{\underline{\mathcal{F}}}(X, Y) \cong D \text{Hom}_{\underline{\mathcal{F}}}(Y, \Sigma^2(X))$$

bifunctorial in X and Y , where $D := \text{Hom}(\cdot, k)$ is the duality with respect to k . Notice that every stably 2-CY Frobenius category is Ext^1 -finite (*i.e.* for every pair of objects X and Y of \mathcal{F} , the dimension of $\text{Ext}_{\mathcal{F}}^1(X, Y)$ is finite).

A *cluster-tilting subcategory* of \mathcal{F} is a full additive subcategory $\mathcal{T} \subset \mathcal{F}$ which is stable under taking direct factors and such that

- (i) for each object X of \mathcal{F} , the functors $\mathcal{F}(X, ?)|_{\mathcal{T}} : \mathcal{T} \rightarrow \text{mod } k$ and $\mathcal{F}(?, X) : \mathcal{T}^{op} \rightarrow \text{mod } k$ are finitely generated;
- (ii) an object X of \mathcal{F} belongs to \mathcal{T} if and only if we have $\text{Ext}_{\mathcal{F}}^1(T, X) = 0$ for all objects T of \mathcal{T} .

A *cluster-tilting object* is a basic object T of \mathcal{F} such that $\mathbf{add}(T)$ is a cluster-tilting subcategory. Equivalently, an object T is cluster-tilting if it is rigid and if each object X satisfying $\mathbf{Ext}_{\mathcal{F}}^1(T, X) = 0$ belongs to $\mathbf{add}(T)$.

An ice quiver consist of a pair (Q, f) , where Q is a quiver together with a distinguished subset $f \subset Q_0$ of frozen vertices such that there are no arrows between any two frozen vertices. We denote by $Q_{\mathcal{X}}$ the quiver of an additive subcategory \mathcal{X} of \mathcal{F} . We can consider $Q_{\mathcal{X}}$ as an ice quiver by considering the vertices corresponding to the projective objects to be frozen and omitting the arrows between them. In this case we will write $Q_{\mathcal{X}}^{fr}$.

Definition 28 ([5]). Let \mathcal{F} be an \mathbf{Ext}^1 -finite Frobenius category whose stable category is 2-Calabi-Yau. The cluster-tilting subcategories of \mathcal{F} determine a cluster structure on \mathcal{F} if the following hold:

- 0) There is at least one cluster-tilting subcategory in \mathcal{F} .
- 1) For each cluster-tilting subcategory \mathcal{T} of \mathcal{F} and each non projective indecomposable X of \mathcal{T} , there is a non projective indecomposable X^* , unique up to isomorphism and not isomorphic to X , such that the additive subcategory $\mathcal{T}' = \mu_X(\mathcal{T})$ of \mathcal{F} with set of indecomposables

$$\mathbf{ind}(\mathcal{T}') = \mathbf{ind}(\mathcal{T}) \setminus \{X\} \cup \{X^*\}$$

is a cluster-tilting subcategory;

- 2) In the situation of 1), there are conflations

$$0 \longrightarrow X^* \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X \xrightarrow{s} E' \xrightarrow{t} X^* \longrightarrow 0,$$

where g and t are minimal right $\mathbf{add}(\mathcal{T} \setminus \{X\})$ -approximations and f and s are minimal left $\mathbf{add}(\mathcal{T} \setminus \{X\})$ -approximations. We call these sequences the *exchange conflations associated to \mathcal{T} and \mathcal{T}'* .

- 3) For any cluster-tilting subcategory \mathcal{T} , the quiver $Q_{\mathcal{T}}^{fr}$ does not have loops nor 2-cycles.
- 4) For each cluster-tilting subcategory \mathcal{T} of \mathcal{F} and each non projective indecomposable X of \mathcal{T} we have that $Q_{\mu_X(\mathcal{T})}^{fr}$ and $\mu_X(Q_{\mathcal{T}}^{fr})$ are related by quiver mutation, i.e. $Q_{\mu_X(\mathcal{T})}^{fr} = \mu_X(Q_{\mathcal{T}}^{fr})$

Definition 29. A Frobenius category \mathcal{E} is a 2-Calabi-Yau realization of the cluster algebra associated to an ice quiver (Q, f) such that

- \mathcal{E} has a cluster structure in the sense of [5].
- \mathcal{E} has a cluster tilting object T , such that the ice quiver of $\mathbf{add}(T)$ is (Q, f) .

The fact that part (i) of Theorem 21 holds, is a direct consequence of Theorem II.1.6 of [5].

5.2. Universal coefficients for finite type quivers. Let us recall the construction of the finite-type cluster algebras with universal coefficients.

Definition 30. Let \mathcal{A} and $\overline{\mathcal{A}}$ be cluster algebras of the same rank n over the coefficient semifields \mathbb{P} and $\overline{\mathbb{P}}$, respectively, with the respective family of cluster variables $(x_{i,t})$ and $(\overline{x}_{i,t})$. We say that $\overline{\mathcal{A}}$ is obtained from \mathcal{A} by a *coefficient specialization* if:

- (i) \mathcal{A} and $\overline{\mathcal{A}}$ have the same exchange matrices at each vertex $t \in \mathbb{T}_n$
- (ii) There is a unique homomorphism of multiplicative groups $\varphi : \mathbb{P} \rightarrow \overline{\mathbb{P}}$ that extends to a (unique) ring homomorphism $\varphi : \mathcal{A} \rightarrow \overline{\mathcal{A}}$ such that $\varphi(x_{i,t}) = \overline{x}_{i,t}$ for all i and all t .

We call φ a coefficient specialization.

Remark 31. By Proposition 12.2 of [11] we know that φ is a coefficient specialization if and only if $\varphi(y_{i,t}) = \overline{y_{i,t}}$ and $\varphi(y_{i,t} \oplus 1) = \overline{y_{i,t}} \oplus 1$ for all $y_{i,t}$. Here $t \mapsto (\mathbf{y}_t, B_t)$ (resp. $t \mapsto (\overline{\mathbf{y}}_t, B_t)$) is the underlying Y -pattern for \mathcal{A} (resp. $\overline{\mathcal{A}}$).

Definition 32. We say that a cluster algebra \mathcal{A} has *universal coefficients* if every cluster algebra with the same family of exchange matrices is obtained from \mathcal{A} by a unique coefficient specialization.

The existence of a cluster algebra with universal coefficients is by no means clear. If it exists, then it can be regarded as an invariant of the mutation class of a quiver. In [11] the authors gave an explicit realization of a universal cluster algebra for any cluster-finite quiver. In a series of papers [25, 26, 27] Reading studied the existence of universal coefficients for cluster algebras beyond finite-type. We recall their construction in the following lines.

Assumption 33. From now on we let Δ be a simply-laced Dynkin diagram and Q a bipartite orientation of Δ .

Definition 34. Let Φ^Δ be the root system associated to Δ . We fix a set of simple roots $\{\alpha_1, \dots, \alpha_n\}$ and denote by $\Phi_{\geq -1}$ the set of *almost positive roots*, i.e. the union of the set of positive roots $\Phi_{>0}$ and the set of negative simple roots. Let $\varepsilon : Q_0 \rightarrow \{+, -\}$ be the function associating a *sign* to each vertex, i.e.

$$\varepsilon(i) = \begin{cases} + & \text{if } i \text{ is a source,} \\ - & \text{if } i \text{ is a sink.} \end{cases}$$

Let $\mathbb{P}_\Delta^{\text{univ}} := \text{Trop}(p_\alpha : \alpha \in \Phi_{\geq -1})$ be the tropical semifield whose generators p_α are labeled by the set $\Phi_{\geq -1}$.

Theorem 35. ([11, Theorem 12.4]) *Let Q be a bipartite orientation of a simply laced Dynkin diagram Δ . Let $\mathcal{A}_Q^{\text{univ}}$ be the cluster algebra with coefficients defined over $\mathbb{P}_\Delta^{\text{univ}}$ whose initial quiver is Q and whose n -tuple of initial coefficients is (y_1, \dots, y_n) where*

$$(5.1) \quad y_j = \prod_{\alpha \in \Phi_{\geq -1}} p_\alpha^{\varepsilon(j)[\alpha : \alpha_j]}.$$

Then $\mathcal{A}_Q^{\text{univ}}$ has universal coefficients.

Remark 36. Since $\mathcal{A}_\Delta^{\text{univ}}$ turns out to be a geometric cluster algebra it can be completely described by an ice quiver. The equation (5.1) allows us to describe at once an ice quiver determining $\mathcal{A}_\Delta^{\text{univ}}$.

5.3. Coxeter functors and bipartite dynamics on roots. We keep the hypothesis of the preceding section. Let W be the Weyl group associated to Φ^Δ and $s_i \in W$ be the reflection associated to the simple root α_i . Following [12] we define two bijections $\tau_\pm : \Phi_{\geq -1} \rightarrow \Phi_{\geq -1}$ by the formula

$$(5.2) \quad \tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j \text{ with } \varepsilon(j) = -\varepsilon; \\ t_\varepsilon(\alpha) & \text{otherwise} \end{cases}$$

where

$$(5.3) \quad t_\varepsilon = \prod_{\varepsilon(k)=\varepsilon} s_k.$$

These elements are well defined because the reflections associated to vertices having the same sign mutually commute. It follows from the same fact that each t_ε is an involution which makes it evident that each τ_ε is bijective.

Remark 37. Let $\text{ind}(kQ)$ denote the set of indecomposable right kQ -modules and $\Phi_{>0}$ be the set of positive roots. The function $\text{ind}(kQ) \rightarrow \Phi_{>0}$ given by

$$M \mapsto \sum_{i=1}^n d_i \alpha_i,$$

where $\underline{\dim}(M) = (d_1, \dots, d_n)$ is the dimension vector of M , is a bijection. We denote by M_α the indecomposable module corresponding to $\alpha \in \Phi_{>0}$. Since $\text{ind}(\mathcal{C}_Q) = \text{ind}(kQ) \cup \{\Sigma P_i\}_{i=1}^n$ this bijection can be extended to a bijection $\text{ind}(\mathcal{C}_Q) \rightarrow \Phi_{\geq -1}$ by sending the object $\Sigma P_i = \tau P_i$ to the negative simple root $-\alpha_i$. Thus the suspension functor $\Sigma = \tau : \mathcal{C}_Q \rightarrow \mathcal{C}_Q$ induces a bijection (which we still call) $\tau : \Phi_{\geq -1} \rightarrow \Phi_{\geq -1}$. Moreover, $\tau_- \circ \tau_+ = \tau$.

Notation 38. Let i be a vertex of Q . Denote by $Q(i)$, $I(i)$ and $S(i)$ the projective, injective and simple module associated to i , respectively. We consider the objects of $\text{mod}(kQ)$ as objects in \mathcal{D}_Q via the canonical embedding $\text{mod}(kQ) \rightarrow \mathcal{D}_Q$.

For each vertex i of Q let $C^i : \text{mod}(kQ) \rightarrow \text{mod}(k\mu_i(Q))$ be the *Bernstein-Gelfand-Ponomarev reflection functor* (cf. [2]). The functor C_i induces an autoequivalence

$$C_{\mathcal{D}}^i : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$$

sending the simple module $S(i)$ to $\Sigma^{-\varepsilon(i)} S(i)$. Let $C_{\mathcal{D}}^\varepsilon$ be the composition of all the reflection functors $C_{\mathcal{D}}^i$ such that $\varepsilon(i) = \varepsilon$. The following is a well known result.

Lemma 39. *The functors $C_{\mathcal{D}}^\varepsilon$ satisfy the following properties:*

- $C_{\mathcal{D}}^\varepsilon \circ C_{\mathcal{D}}^\varepsilon \cong \text{Id}$,
- $\Sigma \circ C_{\mathcal{D}}^\varepsilon \cong C_{\mathcal{D}}^\varepsilon \circ \Sigma$,
- $C_{\mathcal{D}}^\varepsilon(P(i)) = \Sigma^{-\varepsilon} I(i)$,
- if α is not a simple root, then $C_{\mathcal{D}}^\varepsilon(M_\alpha) = M_{\tau_\varepsilon(\alpha)}$,

Corollary 40. *Let M_α be the indecomposable kQ -module corresponding to $\alpha \in \Phi_{>0}$. Let i be a vertex of Q . If i is a source then*

$$[\tau_+(\alpha) : \alpha_i] = \dim \text{Ext}_{kQ}^1(M_\alpha, S(i)).$$

If i is a sink of Q then

$$[\tau_-(\alpha) : \alpha_i] = \dim \text{Ext}_{kQ}^{-1}(M_\alpha, S(i)).$$

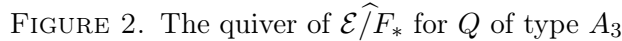
Proof. We know that $[\tau_\varepsilon(\alpha) : \alpha_i] = \dim \text{Hom}(M_{\tau_\varepsilon(\alpha)}, I(i))$. By Lemma 39 we have the following isomorphisms

$$\begin{aligned} \text{Hom}(M_{\tau_\varepsilon(\alpha)}, I(i)) &\cong \text{Hom}(M_{\tau_\varepsilon(\alpha)}, \Sigma^\varepsilon C_{\mathcal{D}}^\varepsilon(P(i))) \\ &= \text{Hom}(C_{\mathcal{D}}^\varepsilon(M_{\tau_\varepsilon(\alpha)}), \Sigma^\varepsilon P(i)) \\ &= \text{Hom}(M_{\tau_\varepsilon \circ \tau_\varepsilon(\alpha)}, \Sigma^\varepsilon P(i)). \\ &= \text{Hom}(M_\alpha, \Sigma^\varepsilon P(i)). \end{aligned}$$

If i is a source then $P(i) = S(i)$. If i is a sink then $P(i) = S(i)$. The claim follows. \square

Example 41. Let $Q : 1 \rightarrow 2 \leftarrow 3$ be a bipartite quiver of type A_3 . There are two τ -orbits in $\Phi_{\geq -1}$ that can be visualized as follows:

In Figure 2 below we have represented the AR quiver of the category $\widehat{\mathcal{E}/F}_*$. Its vertices are labeled by the rule described above.


$$H : \mathcal{R}/\langle \sigma(\mathbb{Z}Q) \rangle \xrightarrow{\sim} \text{ind}(\mathcal{D}_Q).$$
$$x_{\mathcal{D}}^{\wedge} = (\mathcal{R}/\langle \sigma(\mathbb{Z}Q) \rangle)(?, x) \cong \mathcal{D}_Q(H(?), H(x))$$
$$P(x) = \bigoplus_{y \in \mathbb{Z}Q} \mathcal{D}_Q(H(y), H(x)) \otimes \sigma(y)^\wedge$$

where the sum ranges on the set of non-frozen vertices y of $\mathbb{Z}\tilde{Q}$ (or equivalently, each vertex $y \in \mathbb{Z}Q$). In other words, the multiplicity of the projective module $\sigma^\wedge(y)$ in $P(x)$ equals the dimension of the vector space $\mathcal{D}_Q(H(y), H(x))$. Recall the following fact stated in [20].

Theorem 42. ([20, Theorem 3.7]) *For each non-frozen vertex $x \in \mathbb{Z}Q$ we have a projective resolution of \mathcal{R} -modules*

$$(5.4) \quad 0 \rightarrow (\Sigma^{-1}x)^\wedge \rightarrow P(x) \rightarrow x^\wedge \rightarrow x_{\mathcal{D}}^\wedge \rightarrow 0.$$

Corollary 43. *There is a short exact sequence*

$$(5.5) \quad 0 \rightarrow \Sigma^{-1}p(x^\wedge) \rightarrow p(P(x)) \rightarrow p(x^\wedge) \rightarrow 0$$

in $\mathcal{E}/\widehat{F}_*$.

Proof. Notice that $x_{\mathcal{D}}^\wedge$ is supported on the non-frozen vertices of $\mathbb{Z}\tilde{Q}$ and that the rest of the terms in (5.4) belong to \mathcal{E} . It follows that there is a short exact sequence

$$(5.6) \quad 0 \rightarrow \Sigma^{-1}(x^\wedge) \rightarrow P(x) \rightarrow x^\wedge \rightarrow 0$$

in \mathcal{E} . Since the canonical projection $\mathcal{E} \rightarrow \mathcal{E}/\widehat{F}_*$ is exact, the image of (5.6) is exact in $\mathcal{E}/\widehat{F}_*$. \square

The labeling on $\text{ind}(\mathcal{E}/\widehat{F}_*)$ introduced in the preceding section allows us to describe more precisely some of the exchange sequences in $\mathcal{E}/\widehat{F}_*$.

Proposition 44. *Let i be a vertex of Q . The exchange sequences associated to X_{α_i} and $X_{-\alpha_i}$ have the form:*

(i) *If i is a source then*

$$(5.7) \quad 0 \longrightarrow X_{-\alpha_i} \longrightarrow \bigoplus_{i \rightarrow j} X_{-\alpha_j} \oplus P_{-\alpha_i} \longrightarrow X_{\alpha_i} \longrightarrow 0$$

and

$$(5.8) \quad 0 \longrightarrow X_{\alpha_i} \longrightarrow \bigoplus_{\alpha \in \Phi_{>0}} P_{\alpha}^{[\alpha: \alpha_i]} \longrightarrow X_{-\alpha_i} \longrightarrow 0.$$

(ii) *If i is a sink then the exchange conflations have the form*

$$0 \longrightarrow X_{-\alpha_i} \longrightarrow \bigoplus_{\alpha \in \Phi_{>0}} P_{\alpha}^{[\alpha: \alpha_i]} \longrightarrow X_{\alpha_i} \longrightarrow 0$$

and

$$0 \longrightarrow X_{\alpha_i} \longrightarrow \bigoplus_{j \rightarrow i} X_{-\alpha_j} \oplus P_{-\alpha_i} \longrightarrow X_{-\alpha_i} \longrightarrow 0.$$

Proof. Let i be a source of Q and $x \in (\mathbb{Z}\tilde{Q})_0$ be a non-frozen vertex such that

$$p(x^\wedge) = X_{-\alpha_i}.$$

In particular,

$$p(\sigma^{-1}(x)^\wedge) = P_{\tau_-(-\alpha_i)} = P_{-\alpha_i} \quad \text{and} \quad p(\tau^{-1}(x)^\wedge) = X_{\alpha_i}.$$

The mesh relation starting at x gives an exact sequence

$$0 \rightarrow x^\wedge \rightarrow \bigoplus_{x \rightarrow y} y^\wedge \oplus (\sigma^{-1}(x))^\wedge \rightarrow (\tau^{-1}(x))^\wedge \rightarrow 0$$

in \mathcal{E} , where the sum ranges over the set of arrows of $\mathbb{Z}\tilde{Q}$ whose tail is x and whose head is a non-frozen vertex y . By the exactness of p there is an exact sequence

$$(5.9) \quad 0 \rightarrow X_{-\alpha_i} \rightarrow \bigoplus_{x \rightarrow y} p(y^\wedge) \oplus P_{-\alpha_i} \rightarrow X_{\alpha_i} \rightarrow 0$$

in \mathcal{E}/\widehat{F} . By construction, the image of (5.9) in the stable category $\underline{\mathcal{E}/\widehat{F}}_* \cong \mathcal{D}_Q/F$ is the image of the mesh relation in \mathcal{D}_Q starting at $\Sigma P(i)$. Thus

$$\bigoplus_{x \rightarrow y} p(y^\wedge) = \bigoplus_{i \rightarrow j \in Q_1} \Sigma P(j).$$

Since $\Sigma P(j)$ corresponds to $-\alpha_j$ under the bijection $\text{ind}(\mathcal{C}_Q) \rightarrow \Phi_{\geq -1}$ the sequence (5.9) identifies with the sequence (5.7). We claim that (5.8) corresponds to (5.5). Indeed: since

$$\Sigma^{-1}p(x^\wedge) = \tau^{-1}p(x^\wedge)$$

in $\mathcal{E}/\widehat{F}_*$ then (5.5) takes the form

$$0 \rightarrow X_{\alpha_i} \rightarrow \bigoplus_{y \in \mathbb{Z}Q} \mathcal{D}_Q(H(y), H(x)) \otimes p(\sigma(y)^\wedge) \rightarrow X_{-\alpha_i} \rightarrow 0.$$

We have seen that $H(x) = \Sigma P(i) = \Sigma S(i)$. We obtain the following isomorphisms

$$(5.10) \quad \begin{aligned} \bigoplus_{y \in \mathbb{Z}Q} \mathcal{D}_Q(H(y), H(x)) &= \bigoplus_{\alpha \in \Phi_{>0}} \mathcal{D}_Q(M_\alpha, \Sigma S(i)) \\ &= \bigoplus_{\alpha \in \Phi_{>0}} \text{Ext}_{kQ}^1(M_\alpha, S(i)). \end{aligned}$$

We obtain that the multiplicity of P_α in $P(x)$ equals $\dim \text{Ext}_{kQ}^1(M_{\tau_+(\alpha)}, S_i)$. By Corollary 40 we know that $\dim(\text{Ext}_{kQ}^1(M_{\tau_+(\alpha)}, S_i)) = [\alpha, \alpha_i]$, *i.e.*

$$p(P(x)) = \bigoplus_{\alpha \in \Phi_{>0}} P_\alpha^{[\alpha: \alpha_i]}.$$

The claim follows. We can treat similarly the case where i is a sink. \square

Proof of Theorem 21 (ii). In view of part (i) of Theorem 21 and Remark 36 it is enough to prove that there is a cluster tilting object $T \in \mathcal{E}/\widehat{F}_*$ whose frozen quiver coincides with the initial coefficients described by Fomin and Zelevinsky in Theorem 35. We claim that

$$T = \bigoplus_{i \in Q_0} X_{-\alpha_i} \oplus \bigoplus_{\alpha \in \Phi_{\geq -1}} P_\alpha$$

satisfies this assertion. It is clear that T is a cluster tilting object since $\bigoplus_{i \in Q_0} X_{-\alpha_i}$ is a cluster tilting object in \mathcal{C}_Q . Let $\mathcal{T} = \text{add}(T)$. Denote by i the vertex of $Q_{\mathcal{T}}^F$ corresponding to the indecomposable object $X_{-\alpha_i}$ and by $\boxed{\alpha}$ the frozen vertex of $Q_{\mathcal{T}}^F$ corresponding to P_α , $\alpha \in \Phi_{\geq -1}$. We can describe $Q_{\mathcal{T}}^F$ using the exchange sequences of Proposition 44. If i is a source of Q then there are

- $[\alpha : \alpha_i]$ arrows from $\boxed{\alpha}$ to i for each $\alpha \in \Phi_{>0}$,
- one arrow from i to j for each arrow $i \rightarrow j$ in Q ,
- one arrow from i to the vertex corresponding to $\boxed{-\alpha_i}$.

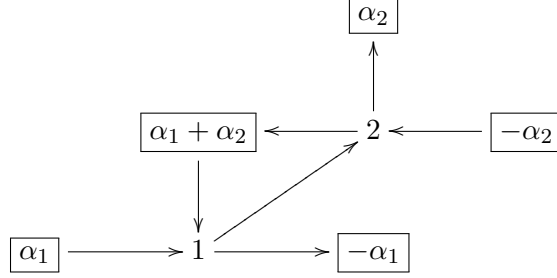
We obtain a similar description when i is a sink of Q . The claim follows. \square

6. AN EXAMPLE

Example 45. Let $Q : 1 \rightarrow 2$ be a Dynkin quiver of type A_2 . Thus, $\varepsilon(1) = +$ and $\varepsilon(2) = -$. By Theorem 35, the cluster algebra with universal coefficients of type A_2 is defined over the tropical semifield

$$\mathbb{P} = \text{Trop}(p_{-\alpha_1}, p_{-\alpha_2}, p_{\alpha_1}, p_{\alpha_1+\alpha_2}, p_{\alpha_2}).$$

The cluster algebra $\mathcal{A}_{A_2}^{\text{univ}}$ is completely determined by the ice quiver

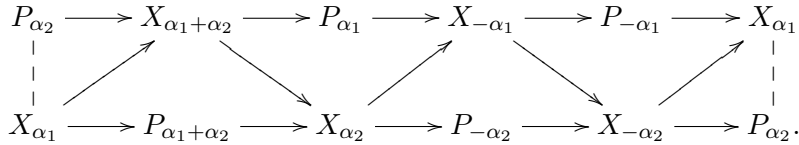


where the frozen vertex corresponding to the generator $\alpha \in \Phi_{\geq -1}$ is denoted by $\boxed{\alpha}$.

Recall that there is a bijection between the cluster variables in $\mathcal{A}_{A_2}^{\text{univ}}$ and the roots in $\Phi_{\geq -1}$. We let x_α denote the cluster variable associated to α . Then the exchange relations in $\mathcal{A}_{A_2}^{\text{univ}}$ are:

$$\begin{aligned} x_{-\alpha_2} x_{\alpha_2} &= p_{-\alpha_2} x_{-\alpha_1} + p_{\alpha_2} p_{\alpha_1+\alpha_2}, \\ x_{-\alpha_1} x_{\alpha_1+\alpha_2} &= p_{\alpha_1} x_{\alpha_2} + p_{-\alpha_1} p_{\alpha_2}, \\ x_{\alpha_2} x_{\alpha_1} &= p_{\alpha_1+\alpha_2} x_{\alpha_1+\alpha_2} + p_{-\alpha_1} p_{-\alpha_2}, \\ x_{\alpha_1+\alpha_2} x_{-\alpha_2} &= p_{\alpha_2} x_{\alpha_1} + p_{-\alpha_2} p_{\alpha_1}, \\ x_{\alpha_1} x_{-\alpha_1} &= p_{-\alpha_1} x_{-\alpha_2} + p_{\alpha_1} p_{\alpha_1+\alpha_2}. \end{aligned}$$

In this case, the quiver of $\widehat{\mathcal{E}}/F$ is the following



We can verify that the exchange relations above correspond to the exchange sequences in $\widehat{\mathcal{E}}/F$. For instances, the exchange relation

$$x_{\alpha_2} x_{\alpha_1} = p_{\alpha_1+\alpha_2} x_{\alpha_1+\alpha_2} + p_{-\alpha_1} p_{-\alpha_2}$$

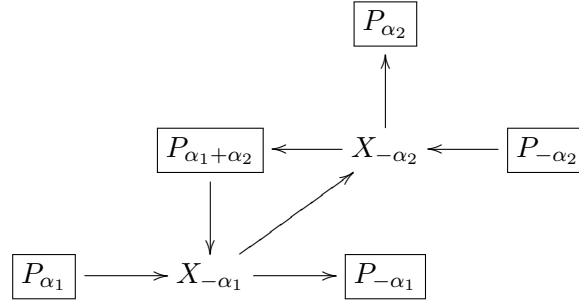
corresponds to the non-split short exact sequences

$$0 \longrightarrow X_{\alpha_1} \longrightarrow P_{\alpha_1+\alpha_2} \oplus X_{\alpha_1+\alpha_2} \longrightarrow X_{\alpha_2} \longrightarrow 0$$

and

$$0 \longrightarrow X_{\alpha_2} \longrightarrow P_{-\alpha_1} \oplus P_{-\alpha_2} \longrightarrow X_{\alpha_1} \longrightarrow 0.$$

The ice quiver of $T = X_{-\alpha_1} \oplus X_{-\alpha_2} \oplus P_{-\alpha_1} \oplus P_{-\alpha_2} \oplus P_{\alpha_1} \oplus P_{\alpha_1+\alpha_2} \oplus P_{\alpha_2}$ is



as expected.

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY.
E-mail address: alfredo_najera@mpim-bonn.mpg.de